

# GROWTH GAP VS. SMOOTHNESS FOR DIFFEOMORPHISMS OF THE INTERVAL

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ABSTRACT. Given a diffeomorphism of the interval, consider the uniform norm of the derivative of its  $n$ -th iteration. We get a sequence of real numbers called the growth sequence. Its asymptotic behavior is an invariant which naturally appears both in smooth dynamics and in geometry of the diffeomorphisms groups. We find sharp estimates for the growth sequence of a given diffeomorphism in terms of the modulus of continuity of its derivative. These estimates extend previous results of Polterovich and Sodin, and Borichev.

## 1. INTRODUCTION AND MAIN RESULTS

Denote by  $\text{Diff}_0[0, 1]$  the group of all  $C^1$ -smooth diffeomorphisms of the interval  $[0, 1]$  fixing the end points 0 and 1. For any  $f \in \text{Diff}_0[0, 1]$ , we define the growth sequence of  $f$  by

$$\Gamma_n(f) = \max\{\|(f^n)'(x)\|_\infty, \|(f^{-n})'(x)\|_\infty\},$$

for all  $n \in \mathbb{N}$ , where  $\|\cdot\|_\infty$  stands for the uniform norm.

We say that a subgroup  $G \subseteq \text{Diff}_0[0, 1]$  admits a *growth gap* if there exists a sequence of positive numbers  $\gamma_n(G)$  that grows sub-exponentially to  $+\infty$ , such that for any  $f \in G$ , either  $\Gamma_n(f)$  tends exponentially to  $+\infty$ , or  $\Gamma_n(f) \leq C(f) \cdot \gamma_n(G)$ , for all  $n \in \mathbb{N}$ .

From a viewpoint of dynamics, growth sequence of an element reflects how the length changes asymptotically under iterations. At the same time, geometrically, growth sequence indicates how an element is distorted with respect to the multiplicative norm. In [DG], D'Ambra and Gromov suggested to study growth sequences of various classes of diffeomorphisms.

The growth sequence is always submultiplicative:

$$\Gamma_{m+n}(f) \leq \Gamma_m(f) \cdot \Gamma_n(f),$$

for all  $m, n \in \mathbb{N}$ . Therefore, the limit

$$\gamma(f) = \lim_{n \rightarrow \infty} \sqrt[n]{\Gamma_n(f)}$$

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always exists. Using standard arguments of ergodic theory, one can check that

$$\gamma(f) = 1 \text{ if and only if } f'(\xi) = 1 \text{ for every } \xi \in \text{Fix}(f),$$

(see [PS], page 199). The following theorem shows that the whole group  $\text{Diff}_0[0, 1]$  does not admit a growth gap (see [B]),

**Theorem 1.** *Given any monotone decreasing sequence of positive numbers  $\{\alpha_n\}_{n=1}^{\infty}$  tending to 0, there exists  $f \in \text{Diff}_0[0, 1]$  such that  $\text{Fix}(f) = \{0, 1\}$ ,  $\gamma(f) = 1$  and*

$$\Gamma_n(f) \geq e^{\alpha_n \cdot n}$$

for all  $n \in \mathbb{N}$ .

As it is shown in Theorem 1, decrease of smoothness assumptions leaves more room for exponential growth, i.e., the growth sequence  $\Gamma_n(f)$  becomes bigger, or in other words, "the growth gap" is smaller. Therefore, smaller subgroups of  $\text{Diff}_0[0, 1]$  should be considered in order to discover a growth gap. In [PS] a growth gap was found for the subgroup of  $C^2$ -diffeomorphisms of  $\text{Diff}_0[0, 1]$ . Namely,

**Theorem 2.** *Let  $f \in \text{Diff}_0[0, 1]$  be a  $C^2$ -diffeomorphism with  $\gamma(f) = 1$ . Then*

$$\Gamma_n(f) \leq C(f) \cdot n^2,$$

for all  $n \in \mathbb{N}$ .

This result leads to a natural question on the growth gap for subgroups of  $\text{Diff}_0[0, 1]$  with intermediate smoothness rate between  $C^1$  and  $C^2$ . A partial answer is provided in [B] and [W]. To introduce this result, we consider the following subgroup of  $\text{Diff}_0[0, 1]$  which is associated with the Hölder condition,  $H_\alpha[0, 1] = \{f \in \text{Diff}_0[0, 1] : |f'(x) - f'(y)| \leq C(f) \cdot |x - y|^\alpha\}$ , for  $0 < \alpha < 1$ .

**Theorem 3.** *If  $f \in H_\alpha[0, 1]$  with  $\gamma(f) = 1$ , then*

$$\log \Gamma_n(f) \leq C(f, \alpha) \cdot n^{1-\alpha},$$

for all  $n \in \mathbb{N}$ .

In the present work we obtain a growth gap for the following intermediate subgroups of diffeomorphisms:

*Case(a):* Subgroups between  $C^2[0, 1] \cap \text{Diff}_0[0, 1]$  and  $\cap_{0 < \alpha < 1} H_\alpha[0, 1]$ .

*Case(b):* Subgroups between  $\text{Diff}_0[0, 1]$  and  $\cup_{0 < \alpha < 1} H_\alpha[0, 1]$ .

To describe subgroups of smoothness between  $C^1$  and  $C^2$ , we use the terminology of moduli of continuity, i.e., non-decreasing continuous

functions  $\omega : [0, 1] \rightarrow \mathbb{R}$  satisfying  $\omega(0) = 0$  and  $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ . Given a modulus of continuity  $\omega : [0, 1] \rightarrow \mathbb{R}_+$ , we consider the subgroup

$$\text{Diff}_0^\omega[0, 1] = \{f \in \text{Diff}_0[0, 1] : \omega_{f'}(\delta) \leq C(f) \cdot \omega(\delta)\},$$

where  $\omega_{f'}(\delta) = \max_{|x-y| \leq \delta} |f'(x) - f'(y)|$ .

It is not hard to check that  $\text{Diff}_0^\omega[0, 1]$  is a non-empty subgroup. Indeed, the identity map is an element of  $\text{Diff}_0^\omega[0, 1]$ . Furthermore, for any two  $f, g \in \text{Diff}_0^\omega[0, 1]$ ,

$$\begin{aligned} & |(f \circ g)(x) - (f \circ g)(y)| \\ & \leq |f'(g(x))g'(x) - f'(g(x))g'(y)| + |f'(g(x))g'(y) - f'(g(y))g'(y)| \\ & \leq A(f) \cdot |g'(x) - g'(y)| + B(f) \cdot |f'(g(x)) - f'(g(y))| \leq C(f, g) \cdot \omega(|x - y|). \end{aligned}$$

$$\begin{aligned} |(f^{-1})'(x) - (f^{-1})'(y)| &= \left| \frac{(f^{-1})'(x) - (f^{-1})'(y)}{(f^{-1})'(x) \cdot (f^{-1})'(y)} \right| \leq \left| \frac{(f^{-1})'(x) - (f^{-1})'(y)}{a(f)^2} \right| \\ &\leq B(f) \cdot \omega(|f^{-1}(x) - f^{-1}(y)|) \leq C(f) \cdot \omega(|x - y|). \end{aligned}$$

Our first result generalizes Theorems 2 and 3 and provides a growth gap for case (a).

**Theorem 4.** *Let  $\omega(x) : [0, 1] \rightarrow \mathbb{R}_+$  be a strictly increasing modulus of continuity. Then, for each  $f \in \text{Diff}_0^\omega[0, 1]$ , such that  $\gamma(f) = 1$ , we have*

$$(*) \quad \log \Gamma_n(f) \leq \log \frac{n}{\omega^{-1}(\frac{2}{n})} + C(f)n\omega(\frac{1}{n}).$$

Here we denote by  $\omega^{-1}$  the inverse function to  $\omega$ .

One can substitute  $\omega(\delta) = \delta$  and  $\omega(\delta) = \delta^\alpha$  into Theorem 4 for achieving Theorems 2 and 3. In the following corollary, we consider two toy models related to *case(a)* in order to test how Theorem 4 provides a growth gap. In the case when the modulus of continuity  $\omega(\delta)$  is close to the identity, the second term on the right hand side of (\*) can be absorbed into the first one. Namely,

**Corollary 1.** (1) *If*

$$\limsup_{x \rightarrow 0} \frac{\omega(x)}{x \cdot \log \frac{e}{x}} < +\infty,$$

*then*

$$\log \Gamma_n(f) \leq C(f, \omega) \cdot \log \frac{n}{\omega^{-1}(\frac{2}{n})}.$$

(2) If

$$\lim_{x \rightarrow 0} \frac{\omega(x)}{x \cdot \log \frac{\varepsilon}{x}} = 0,$$

then

$$\log \Gamma_n(f) \leq (1 + o(1)) \cdot \log \frac{n}{\omega^{-1}(\frac{2}{n})}.$$

The proofs easily follow by substituting the relevant assumptions into Theorem 4.

The drawback of Theorem 4 is that it does not provide a growth gap for case (b). For instance, if we consider a diffeomorphism  $f(x)$  from case (b) with  $\omega_f(\delta) \leq \frac{1}{\log \frac{\varepsilon}{\delta}}$ , then an attempt to apply Theorem 4 for this diffeomorphism yields only a trivial estimate

$$\log \Gamma_n(f) \leq C(f) \cdot n.$$

Our second theorem mends this disadvantage. It shows that in case (b) (under additional regularity assumption imposed on  $\omega$ ) one can discard the first term on the right hand side of (\*) :

**Theorem 5.** *Let  $\omega(x) : [0, 1] \rightarrow \mathbb{R}_+$  be a modulus of continuity such that for some  $0 < \alpha < 1$ ,  $\frac{\omega(x)}{x^\alpha}$  is a decreasing function on  $(0, a(\alpha))$ , where  $0 < a(\alpha) < 1$ . Then for  $f \in \text{Diff}_0^\omega[0, 1]$ , such that  $\gamma(f) = 1$ , we have*

$$\log \Gamma_n(f) \leq C(f) \cdot n\omega\left(\frac{1}{n}\right).$$

The next set of theorems present a sufficient sharpness for the estimates of the bounds in Theorems 4 and 5 respectively.

**Theorem 6.** *Suppose that for each  $0 < \alpha < 1$  there exists  $0 < a(\alpha) < 1$  such that the function  $\frac{\omega(x)}{x^\alpha}$  increases for all  $x \in [0, a(\alpha)]$  while the function  $\frac{\omega(x)}{x}$  is decreasing for all  $x \in [0, 1]$  and suppose that*

$$\lim_{x \rightarrow 0} \frac{\omega(x)}{x \cdot \log(\frac{\varepsilon}{x})} = 0.$$

*Then, there exists a diffeomorphism  $f \in \text{Diff}_0^\omega[0, 1]$  with  $\gamma(f) = 1$  such that for any  $\varepsilon > 0$ ,*

$$\log \Gamma_n(f) \geq (1 - \varepsilon) \cdot \log \frac{n}{\omega^{-1}(\frac{c(f)}{n})}, \quad n \rightarrow \infty.$$

**Theorem 7.** *Suppose that the modulus of continuity  $\omega$  satisfies assumptions of Theorem 5. Then there exists a diffeomorphism  $f \in \text{Diff}_0^\omega[0, 1]$  with  $\gamma(f) = 1$ , such that for each  $\varepsilon > 0$ ,*

$$\log \Gamma_n(f) \geq c(\varepsilon) n^{1-\varepsilon} \omega\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

The proofs of Theorems 4-7 use ideas and techniques introduced in [L, chapter II] and especially in [B].

## 2. GROWTH GAP: PROOFS OF THEOREMS 4 AND 5

The following lemma (see [EF, Dz]) states that every modulus of continuity admits an equivalent concave modulus of continuity:

**Lemma 1.** *For any modulus of continuity  $\omega$  there exists a concave modulus of continuity  $\omega^*$  such that  $\omega \leq \omega^* \leq 2\omega$  everywhere on  $[0, 1]$ .*

Due to this lemma, we assume in the proofs of Theorems 4 and 5 that  $\omega$  is a concave modulus of continuity.

*Proof of Theorem 4.* First, we will introduce several notations and definitions:  $\phi(x) := f(x) - x$ ;  $x_n = f(x_{n-1})$ ;  $A = \max_{x \in [0, 1]} f'(x)$ ,  $a = \min_{x \in [0, 1]} f'(x)$ . Choose a sufficiently small  $\varepsilon > 0$ , such that we will have  $\omega(\varepsilon) < 1$ . Consider a function  $x \mapsto x \cdot \omega(x)$  which maps  $[0, \varepsilon]$  on  $[0, \varepsilon \cdot \omega(\varepsilon)]$ , and denote by  $\Omega(x) : [0, \varepsilon \cdot \omega(\varepsilon)] \rightarrow [0, \varepsilon]$  its inverse. Now pick a positive  $\delta < \varepsilon$ , so that the following requirement will be satisfied:

For all  $x \in [0, \delta]$  we have  $\phi(x) \in [0, \varepsilon \cdot \omega(\varepsilon)]$  and

$$J_x := [x, f(x)] \subseteq I_x := [x - \Omega(\phi(x)), x + \Omega(\phi(x))] \subseteq [0, \varepsilon].$$

Let us explain why it is possible. It is obviously possible to require that  $\phi(x) \in [0, \varepsilon \cdot \omega(\varepsilon)]$  and  $x + \Omega(\phi(x)) \leq \varepsilon$  for all  $x \in [0, \delta]$ , due to continuity. The inequality

$$0 \leq x - \Omega(\phi(x))$$

is equivalent to that

$$\phi(x) \leq x \cdot \omega(x)$$

which is satisfied for all  $x \in [0, \delta]$ , since  $|\phi'(x)| \leq \omega(x)$ .

We will present now a sequence of technical claims, which will be used later in the proof of Theorem 4.

**Claim 1.** (a) For any  $x \in [0, 1]$  and  $y \in [x, f(x)]$ ,

$$\frac{1}{1+A} \leq \frac{\phi(x)}{\phi(y)} \leq \frac{1}{1+a}.$$

(b) For any  $x_1 \in [0, 1]$  and  $n \in \mathbb{N}$ ,

$$\frac{1}{1+A} \cdot n \leq \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \leq \frac{1}{1+a} \cdot n.$$

*Proof of Claim 1.* For any  $y \in [x, f(x)]$ , there exists  $0 \leq \theta \leq 1$  such that  $y = x + \theta \cdot \phi(x)$  and  $0 \leq \theta_1 \leq 1$ , such that

$$\frac{\phi(y)}{\phi(x)} = \frac{\phi(x) + \theta\phi(x)\phi'(x + \theta_1\theta\phi(x))}{\phi(x)} \leq 1 + \max_{x \in [0,1]} |\phi'(x)| \leq 1 + A.$$

In the same way,

$$\frac{\phi(x)}{\phi(y)} = \frac{\phi(x)}{\phi(x) + \theta\phi(x)\phi'(x + \theta_1\theta\phi(x))} \leq \frac{1}{1 + \phi'(x + \theta_1\theta\phi(x))} \leq \frac{1}{1+a}.$$

Therefore, for all  $k \in \mathbb{N}$ :

$$\frac{1}{1+A} \leq \int_{x_k}^{x_{k+1}} \frac{dt}{\phi(t)} \leq \frac{1}{1+a}.$$

By summing the integrals we obtain the desirable inequality.  $\square$

**Claim 2.** For all  $x \in [0, \delta]$  and  $y \in I_x$ , we have

$$(a) \quad |\phi'(y)| \leq 3\omega(\Omega(\phi(x))) = 3 \frac{\phi(x)}{\Omega(\phi(x))}.$$

$$(b) \quad \frac{\phi(y)}{\phi(x)} \leq 4.$$

**Remark:** In particular, we obtain that for all  $x \in [0, \delta]$ ,  $|\phi'(x)| \leq 3\omega(\Omega(\phi(x)))$ .

*Proof of Claim 2.* (a) Suppose that there exists  $y_0 \in I_x$  such that  $\phi'(y_0) > 3 \cdot \omega(\Omega(\phi(x)))$ . Note that following inequalities are satisfied for all  $y \in I_x$ :

$$\begin{aligned} \phi'(y) &\geq \phi'(y_0) - \omega(|y - y_0|) > 3 \cdot \omega(\Omega(\phi(x))) - \omega(2\Omega(\phi(x))) \\ &\geq (3 - 2) \cdot \omega(\Omega(\phi(x))) = \omega(\Omega(\phi(x))). \end{aligned}$$

Therefore,

$$\phi(x) - \phi(x - \Omega(\phi(x))) = \int_{x - \Omega(\phi(x))}^x \phi'(t) dt > \omega(\Omega(\phi(x))) \cdot \Omega(\phi(x)) = \phi(x).$$

It follows that  $\phi(x - \Omega(\phi(x))) < 0$ , what contradicts our assumptions. Now, assume that there exists a point  $y_0 \in I_x$  such that  $\phi'(y_0) < -3 \cdot \omega(\Omega(\phi(x)))$ . Then for all  $y \in I_x$ ,

$$\begin{aligned} \phi'(y) &\leq \phi'(y_0) + \omega(|y - y_0|) < -3 \cdot \omega(\Omega(\phi(x))) + \omega(2\Omega(\phi(x))) \\ &\leq (-3 + 2)\omega(\Omega(\phi(x))) \leq -\omega(\Omega(\phi(x))). \end{aligned}$$

Therefore,

$$\phi(x + \Omega(\phi(x))) - \phi(x) = \int_x^{x + \Omega(\phi(x))} \phi'(t) dt < -\omega(\Omega(\phi(x))) \cdot \Omega(\phi(x)) = -\phi(x),$$

whence  $\phi(x + \Omega(\phi(x))) < 0$ , this is a contradiction.

(b) Using (a) we obtain for some  $0 < \theta_1 < 1$ ,

$$\begin{aligned} \frac{\phi(y)}{\phi(x)} &= \frac{\phi(x) + (y - x)\phi'(x + \theta_1(y - x))}{\phi(x)} \leq 1 + \max_{y \in I_x} |\phi'(y)| \cdot \frac{|I_x|}{\phi(x)} \leq \\ &\leq 1 + 3 \cdot \omega(\Omega(\phi(x))) \frac{\Omega(\phi(x))}{\phi(x)} = 4. \end{aligned}$$

□

**Claim 3.** Let  $z \in [0, \delta]$  and  $n \in \mathbb{N}$  be such that

$$n \geq c(f) \cdot \frac{\Omega(z)}{z}.$$

Then,

$$\frac{1}{z} \leq C(f) \cdot \frac{n}{\omega^{-1}(\frac{1}{n})}.$$

*Proof of Claim 3.* Denote by  $s = \Omega(z)$ , and notice that  $s \cdot \omega(s) = z$ . Thus,

$$\begin{aligned} \omega(s) &= \frac{z}{\Omega(z)} \geq \frac{c(f)}{n} \\ s &\geq \omega^{-1}\left(\frac{c(f)}{n}\right) \geq C(f) \cdot \omega^{-1}\left(\frac{1}{n}\right), \end{aligned}$$

therefore,

$$z \geq s \cdot \frac{c(f)}{n} \geq c(f) \cdot C(f) \frac{\omega^{-1}(\frac{1}{n})}{n},$$

and we are done. □

We turn now to the following two lemmas, on which the proof of Theorem 4 will be based.

**Lemma 2.** *Suppose that  $x_1, \dots, x_{n+1} \in (0, \delta)$ . Then,*

$$|\log(\frac{\phi(x_{n+1})}{\phi(x_1)})| \leq \log \frac{n}{\omega^{-1}(\frac{1}{n})}.$$

*Proof of Lemma 2.* We split the proof to 2 cases.

Case 1: a.  $x_{n+1} \in I_{x_1}$  and  $\phi(x_1) < \phi(x_{n+1})$ . In this case,

$$|\log \frac{\phi(x_1)}{\phi(x_{n+1})}| = \log \frac{\phi(x_{n+1})}{\phi(x_1)} < \log 4,$$

due to Claim 2

b.  $x_{n+1} \in I_{x_1}$  and  $\phi(x_1) > \phi(x_{n+1})$ . We have two possibilities:

- (i)  $x_1 + \Omega(\phi(x_{n+1})) > x_{n+1}$  and by Claim 2  $\frac{\phi(x_1)}{\phi(x_{n+1})} < 4$ .
- (ii)  $x_1 + \Omega(\phi(x_{n+1})) \leq x_{n+1}$ , then :

$$\begin{aligned} n &\geq (1+a) \cdot \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \geq (1+a) \cdot \int_{x_{n+1}-\Omega(\phi(x_{n+1}))}^{x_{n+1}} \frac{dt}{\phi(t)} \\ &\geq 4(1+a) \cdot \frac{\Omega(\phi(x_{n+1}))}{\phi(x_{n+1})}. \end{aligned}$$

Hence, this case is completed due to Claim 3

Case 2: a.  $x_{n+1} \notin I_{x_1}$  and  $\phi(x_1) < \phi(x_{n+1})$ .

$$n \geq (1+a) \cdot \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \geq (1+a) \cdot \int_{x_1}^{x_1+\Omega(\phi(x_1))} \frac{dt}{\phi(t)} \geq \frac{4(1+a)}{\phi(x_1)} \cdot \Omega(\phi(x_1)),$$

in the last inequality we have used Claim 1, hence we are done due to Claim 3.

b.  $x_{n+1} \notin I_{x_1}$  and  $\phi(x_1) > \phi(x_{n+1})$ .

$$\begin{aligned} n &\geq \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \geq (1+a) \int_{x_{n+1}-\Omega(\phi(x_1))}^{x_{n+1}} \frac{dt}{\phi(t)} \geq (1+a) \int_{x_{n+1}-\Omega(\phi(x_{n+1}))}^{x_{n+1}} \frac{dt}{\phi(t)} \\ &\geq 4(1+a) \cdot \frac{\Omega(\phi(x_{n+1}))}{\phi(x_{n+1})} \end{aligned}$$

In the last inequality we have used Claim 1, hence we are done due to Claim 3.  $\square$

**Lemma 3.** *Suppose that  $x_1, \dots, x_n \in (0, \delta)$ . Then,*

$$|\log(f^n)'(x_1) - \log(\frac{\phi(x_n)}{\phi(x_1)})| \leq C(f) \cdot n \cdot \omega(\frac{1}{n}).$$



*Proof of Lemma 3.* We have

$$\begin{aligned} |(f^n)'(x_1) - \log \frac{\phi(x_n)}{\phi(x_1)}| &= \left| \sum_{k=1}^{n-1} (\log(1 + \phi'(x_k)) - \log \frac{\phi(x_{k+1})}{\phi(x_k)}) \right| \\ &\leq \sum_{k=1}^{n-1} \left| \int_{x_k}^{x_{k+1}} \frac{\phi'(t)}{\phi(t)} dt - \log(1 + \phi'(x_k)) \right|. \end{aligned}$$

The inequality  $-\frac{y^2}{1+y} \leq \log(1+y) - y < 0$ , which is valid for all  $y > -1$ , implies that  $|\log(1+y) - y| \leq \frac{y^2}{1+y}$ . In our context, we may use both inequalities, since  $\min_{x \in [0,1]} \phi'(x) > -1$ .

$$\begin{aligned} &\sum_{k=1}^{n-1} \left| \int_{x_k}^{x_{k+1}} \frac{\phi'(t)}{\phi(t)} dt - \log(1 + \phi'(x_k)) \right| \leq \\ &\sum_{k=1}^{n-1} \left| \int_{x_k}^{x_{k+1}} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x_k) \right| + \sum_{k=1}^{n-1} |\log(1 + \phi'(x_k)) - \phi'(x_k)| \\ &\leq \sum_{k=1}^{n-1} \left| \int_{x_k}^{x_{k+1}} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x_k) \right| + \sum_{k=1}^{n-1} \frac{[\phi'(x_k)]^2}{1 + \phi'(x_k)} \\ &\leq \sum_{k=1}^{n-1} \left| \int_{x_k}^{x_{k+1}} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x_k) \right| + \frac{1}{1+a} \cdot \sum_{k=1}^{n-1} [\phi'(x_k)]^2. \end{aligned}$$

Now we are going to estimate these sums. For any  $x \in [0, \delta]$ , there exists  $0 < \theta < 1$  such that

$$\begin{aligned} \int_x^{x+\phi(x)} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x) &= \frac{\phi'(x + \theta \cdot \phi(x))}{\phi(x + \theta \cdot \phi(x))} \cdot \phi(x) - \phi'(x) = \\ &= [\phi'(x + \theta \phi(x)) - \phi'(x)] \cdot \frac{\phi(x)}{\phi(x + \theta \phi(x))} + \phi'(x) \left[ \frac{\phi(x)}{\phi(x + \theta \phi(x))} - 1 \right]. \end{aligned}$$

By Claim 1,

$$\begin{aligned} &|[\phi'(x + \theta \phi(x)) - \phi'(x)] \cdot \frac{\phi(x)}{\phi(x + \theta \phi(x))}| \\ &\leq (|\phi'(x + \theta \phi(x)) - \phi'(x)|) \cdot \max_{y \in J_x} \frac{\phi(x)}{\phi(y)} \\ &\leq (1+A) \cdot \omega(\theta \cdot \phi(x)) \leq (1+A) \cdot \omega(\phi(x)). \end{aligned}$$

Then, there exists some  $0 < \theta_1 < 1$ , such that:  $\phi(x + \theta \cdot x) - \phi(x) = \theta \cdot x \cdot \phi'(x + \theta \cdot \theta_1 \cdot x)$ . Using it together with Claim 1, we get

$$\left| \frac{\phi(x)}{\phi(x + \theta \phi(x))} - 1 \right| = \left| \frac{\phi(x)}{\phi(x) + \theta \phi(x) \phi'(x + \theta_1 \theta \phi(x))} - 1 \right| =$$

$$\begin{aligned}
&= \left| \frac{1}{1 + \theta\phi'(x + \theta_1\theta\phi(x))} - 1 \right| = \left| \frac{\theta\phi'(x + \theta_1\theta\phi(x))}{1 + \theta\phi'(x + \theta_1\theta\phi(x))} \right| \\
&\leq \frac{1}{1+a} \cdot |\phi'(x + \theta_1\theta\phi(x))| \leq \frac{3}{1+a} \cdot \omega(\Omega(\phi(x))).
\end{aligned}$$

Therefore

$$\begin{aligned}
|\phi'(x) \cdot [\frac{\phi(x)}{\phi(x + \theta\phi(x))} - 1]| &\leq \frac{3}{1+a} \cdot |\phi'(x)| \cdot \omega(\Omega(\phi(x))) \\
&\leq \frac{9}{1+a} \cdot \omega^2(\Omega(\phi(x))).
\end{aligned}$$

Since  $\Omega(x) \geq x$ , it follows that  $\Omega(\phi(x)) \geq \phi(x)$ . Additionally,  $\frac{\omega(x)}{x}$  is decreasing, thus

$$\frac{\omega(\Omega(\phi(x)))}{\Omega(\phi(x))} \leq \frac{\omega(\phi(x))}{\phi(x)}.$$

The substitution of it yields the following:

$$\omega^2(\Omega(\phi(x))) = \phi(x) \cdot \frac{\omega(\Omega(\phi(x)))}{\Omega(\phi(x))} \leq \phi(x) \cdot \frac{\omega(\phi(x))}{\phi(x)} = \omega(\phi(x)).$$

Adding those results together, we have the following estimate:

$$\left| \int_x^{x+\phi(x)} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x) \right| \leq (1 + A + \frac{9}{1+a}) \cdot \omega(\phi(x)).$$

Using the previous estimate, we have also:

$$|\phi'(x)|^2 \leq 9 \cdot \omega^2(\Omega(\phi(x))) \leq 9 \cdot \omega(\phi(x)).$$

Let us apply the above estimates for bounding our initial expressions:

$$\begin{aligned}
&\sum_{k=1}^{n-1} \left| \int_{x_k}^{x_{k+1}} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x_k) \right| + C \cdot \sum_{k=1}^{n-1} [\phi'(x_k)]^2 \\
&\leq C(f) \cdot \sum_{k=1}^{n-1} \omega(\phi(x_k)),
\end{aligned}$$

with  $C(f) = 10 + A + \frac{9}{1+a}$ . By Jensen's inequality

$$\sum_{k=1}^{n-1} \omega(\phi(x_k)) = \sum_{k=1}^{n-1} \omega(x_{k+1} - x_k) \leq (n-1) \cdot \omega\left(\frac{1}{n-1}\right),$$

completing the proof of Lemma 2. □

Combining Lemmas 1 and 2, we get

**Corollary 2.** *Suppose that  $x_1, \dots, x_n \in (0, \delta)$ . Then,*

$$(**) \quad |\log (f^n)'(x_1)| \leq \log \frac{n}{\omega^{-1}(\frac{1}{n})} + C(f) \cdot n \cdot \omega(\frac{1}{n}).$$

At last, we turn to the details of the proof of Theorem 4, we shall show that estimate  $(**)$  holds for each  $x \in (0, 1)$ . Consider the decomposition of the interval into a union of open intervals  $[0, 1] \setminus \text{Fix}(f) = \cup_{i \in I} (a_i, b_i)$ . Let  $x \in (0, 1)$  be an arbitrary point, then  $x \in (a_i, b_i)$  for some  $i \in I$ . If  $|b_i - a_i| \leq \delta$ , then the proof is complete by Corollary 2. There are only finitely many intervals such that  $|b_i - a_i| > \delta$ . We take one of them and divide it into 3 subintervals:

$$[a_i, b_i] = [a_i, a_i + \delta_0] \cup [a_i + \delta_0, b_i - \delta_0] \cup [b_i - \delta_0, b_i],$$

when  $\delta_0 \leq \delta$  and  $\Omega(\phi(x)) \in [a_i, b_1 - \delta_0]$  for all  $x \in [a_i, a_i + \delta_0]$ . We denote by  $n_1, n_2, n_3$  the length of the trajectory of the sequence  $x_n$  in each of the 3 subintervals respectively.

It is evident that  $n_2$  is bounded by some constant  $N(f)$ . If  $n_3 = 0$  or  $n_1 = 0$ , then we are done due to Corollary 2. Otherwise,  $n = n_1 + n_2 + n_3$ ,

$$|\log (f^n)'(x_1)| \leq |\log (f^{n_2})'(x_{n_1+1})| + |\log (f^{n_1})'(x_1) + \log (f^{n_3})'(x_{n_1+n_2+1})|,$$

we continue using Lemma 2,

$$\leq N(f) \cdot C(f) + |\log \frac{\phi(x_{n_1}) \cdot \phi(x_n)}{\phi(x_1) \cdot \phi(x_{n_1+n_2+1})}| + C(f)n_1\omega(\frac{1}{n_1}) + C(f)n_3\omega(\frac{1}{n_3}).$$

Note that

$$C(f) \cdot n_1\omega(\frac{1}{n_1}) + C(f) \cdot n_3\omega(\frac{1}{n_3}) \leq 2C(f) \cdot \omega(\frac{1}{n}).$$

Moreover, we have the following estimate:

$$|\log \frac{\phi(x_{n_1})}{\phi(x_{n_1+n_2+1})}| \leq c_i = \max_{z \in [f^{-1}(a_i+\delta_0), a_i+\delta_0], w \in [f^{-1}(b_i-\delta_0), b_i-\delta_0]} |\log \frac{\phi(z)}{\phi(w)}|.$$

Now we are going to find an upper bound for  $|\log \frac{\phi(x_n)}{\phi(x_1)}|$ . As before, we split into two cases:

a.  $\phi(x_n) > \phi(x_1)$ . By using Claim 1 and the choice of  $\delta_0$ , we have

$$\begin{aligned} n &\geq (1-a) \cdot \int_{x_1}^{x_{n_1+n_2}} \frac{dt}{\phi(t)} \\ &\geq (1-a) \cdot \int_{x_1}^{x_1+\Omega(\phi(x_1))} \frac{dt}{\phi(t)} \geq \frac{1-a}{4} \cdot \frac{\Omega(\phi(x_1))}{\phi(x_1)}, \end{aligned}$$

the last inequality is due to Claim 2. Thus by Claim 3, we have

$$|\log \frac{\phi(x_n)}{\phi(x_1)}| \leq \log \frac{1}{\phi(x_1)} \leq C(f) \cdot \frac{n}{\omega^{-1}(\frac{1}{n})}.$$

b.  $\phi(x_n) < \phi(x_1)$ . Then,

$$\begin{aligned} n &\geq n_2 + n_3 \geq (1-a) \cdot \int_{x_{n_1}}^{x_n} \frac{dt}{\phi(t)} \\ &\geq (1-a) \cdot \int_{x_n - \Omega(\phi(x_n))}^{x_n} \frac{dt}{\phi(t)} \geq \frac{4}{1-a} \cdot \frac{\Omega(\phi(x_n))}{\phi(x_n)}. \end{aligned}$$

In the same way, by Claim 3 it follows that

$$|\log \frac{\phi(x_n)}{\phi(x_1)}| \leq \log \frac{1}{\phi(x_n)} \leq C(f) \cdot \frac{n}{\omega^{-1}(\frac{1}{n})}.$$

□

*Proof of Theorem 5.* Without limiting the generality, we assume that  $\omega$  is a  $C^1$  smooth concave function. The proof is based on the following idea. First, let us define a function

$$\phi_0(x) = \int_0^x \omega(t) dt,$$

and the corresponding  $f_0(x) = x - \phi_0(x)$ , on some interval  $[0, \varepsilon]$ , where  $\varepsilon$  will be determined soon. Note that  $f_0(0) = 0$ ,  $f'_0(0) = 1 - \omega(0) = 1$ . Moreover,  $|f'_0(x) - f'_0(y)| = |\omega(x) - \omega(y)| \leq \omega(|x - y|)$ . Let us check when  $f_0$  is increasing, we have  $f'_0(x) = 1 - \omega(x)$ , thus we pick  $\varepsilon < \frac{1}{2}$ , such that  $\omega(x) < 1$  for all  $x \in [0, \varepsilon]$ . Further we extend  $f_0$  to the whole interval in the following manner. On the interval  $[1 - \varepsilon, 1]$ ,  $f_0$  consists of an symmetric copy of  $f_0(x)$  which is defined on  $[0, \varepsilon]$ , in a way that  $f_0(1) = 1$ ,  $f'_0(1) = 1$ . The middle interval  $[\varepsilon, 1 - \varepsilon]$  is inessential, we extend there  $f_0$  to be an arbitrary  $C^\infty$  function.

By Lemma 2,

$$\log(f_0^n)'(x_1) \leq \log \frac{\phi_0(x_1)}{\phi_0(x_{n+1})} + C \cdot n \cdot \omega\left(\frac{1}{n}\right).$$

First, we will show that  $\log \frac{\phi_0(x_1)}{\phi_0(x_{n+1})} \leq C \cdot n \cdot \omega\left(\frac{1}{n}\right)$ , this will prove Theorem 5 for  $f_0$ . Then, using Lemma 5 below, we shall that this bound holds for our original function  $f \in \text{Diff}_0^\omega[0, 1]$ , perhaps, with a bigger constant  $C$  on the right hand side.

In this section,  $\{x'_n\}_{n \in \mathbb{N}}$  will denote the iterations of  $f_0$ , i.e., for given  $x'_0 \in (0, 1)$ , we denote by  $x'_{n+1} = f_0(x'_n)$  for all  $n \in \mathbb{N}$ .

**Lemma 4.** *The following inequality is satisfied*

$$\log \frac{\phi_0(x'_0)}{\phi_0(x'_n)} \leq C \cdot n \cdot \omega\left(\frac{1}{n}\right).$$

*Proof of Lemma 4.* We will prove it only for  $x'_0, \dots, x'_n \in (0, \varepsilon)$ , the general case is proved similarly. Denote by  $s := s(n) = \log(\frac{\phi_0(x'_0)}{\phi_0(x'_n)})$ . We wish to show that  $s \leq C \cdot n \omega(\frac{1}{n})$ . We have the following inequality, which is valid due to the Claim 1:  $n := n(s) \geq C' \cdot \int_{x'_n}^{x'_0} \frac{dt}{\phi_0(t)}$ . We are going to change variables in this integral until we bound  $n(s)$  from above by some function of  $s$ .

$$\begin{aligned} n &\geq C' \cdot \int_{x'_n}^{x'_0} \frac{dt}{\phi_0(t)} = C' \cdot \int_{\phi_0^{-1}(e^{-s}\phi_0(x_0))}^{x'_0} \frac{dt}{\phi_0(t)} = [t = \phi_0^{-1}(w)] = \\ &= C' \cdot \int_{e^{-s} \cdot \phi_0(x_0)}^{\phi_0(x_0)} \frac{dw}{w \cdot \phi'_0(\phi_0^{-1}(w))} = [w = e^{-u}] = C' \cdot \int_{\log(\frac{1}{\phi_0(x_0)})}^{s + \log(\frac{1}{\phi_0(x_0)})} \frac{du}{\phi'_0(\phi_0^{-1}(e^{-u}))} = \\ &= [t = u - \log(\frac{1}{\phi_0(x_0)})] = C' \cdot \int_0^s \frac{dt}{\omega(\phi_0^{-1}(e^{-t} \cdot \phi_0(x_0)))}. \end{aligned}$$

We obtain that  $n \geq C' \cdot \int_0^s \frac{dt}{\omega(\phi_0^{-1}(e^{-t} \cdot \phi_0(x_0)))}$ . The inequality  $s \leq C \cdot n \omega(\frac{1}{n})$  will be proved if we find an absolute constant  $C$  such that

$$C' \cdot \int_0^{C \cdot n \omega(\frac{1}{n})} \frac{dt}{\omega(\phi_0^{-1}(e^{-t} \cdot \phi_0(x_0)))} \geq n.$$

Denote by  $\Lambda(x) = x \cdot \omega(\frac{1}{x})$ , for  $x \geq 1$  and continue  $\Lambda(x)$  in an arbitrary monotonically and smooth way on  $[0, 1]$ , such that  $\Lambda(0) = 0$ . Let  $\xi := \Lambda(n)$ ,  $n = \Lambda^{-1}(\xi)$ . Note that

$$\xi = \frac{\omega(\frac{1}{n})}{\frac{1}{n}} = \frac{\omega(\frac{1}{n})}{(\frac{1}{n})^\alpha} \cdot n^{1-\alpha} \rightarrow \infty,$$

as  $n \rightarrow \infty$ , since  $x^{-\alpha} \omega(x)$  is decreasing.

Thus, we have to show that, for  $\xi \rightarrow \infty$ ,

$$C' \cdot \int_0^{C \cdot \xi} \frac{dt}{\omega(\phi_0^{-1}(e^{-t} \cdot \phi_0(x_0)))} \geq \Lambda^{-1}(\xi).$$

Making once again change of variables  $[t = C \cdot s]$ , we get

$$C' \cdot \int_0^\xi \frac{C \cdot ds}{\omega(\phi_0^{-1}(e^{-C \cdot s} \cdot \phi_0(x_0)))} \geq \Lambda^{-1}(\xi).$$

This is equivalent to

$$\int_0^\xi \frac{C' \cdot C}{\omega(\phi_0^{-1}(e^{-C \cdot t} \cdot \phi_0(x_0)))} - (\Lambda^{-1}(t))' dt \geq 0.$$

In order to prove that inequality let us show that the expression inside the integral is always non-negative. I.e., we have to show that

$$\frac{C \cdot C'}{\omega(\phi_0^{-1}(e^{-C \cdot t} \cdot \phi_0(x_0)))} \geq \frac{1}{\Lambda'(\Lambda^{-1}(t))},$$

or,

$$C \cdot C' \cdot \Lambda'(\Lambda^{-1}(t)) \geq \omega(\phi_0^{-1}(e^{-t \cdot C} \cdot \phi_0(x_0))).$$

**Claim 4.** *There exists a constant  $c > 0$ , such that  $\Lambda'(x) \geq c \cdot \omega(\frac{1}{x})$ .*

*Proof of Claim 4.* Indeed,  $\Lambda'(x) = \omega(\frac{1}{x}) - \frac{1}{x} \cdot \omega'(\frac{1}{x})$ , thus we have to show that -

$$0 \geq (c-1)\omega(\frac{1}{x}) + \frac{\omega'(\frac{1}{x})}{x}$$

By multiplying both sides by  $\frac{x}{\omega(\frac{1}{x})}$ , we obtain

$$0 \geq (c-1) \cdot x + \frac{\omega'(\frac{1}{x})}{\omega(\frac{1}{x})}$$

Denoting by  $t = \frac{1}{x}$ , where  $0 < t < 1$ , we have to prove the following inequality:

$$0 \geq (\log(\frac{\omega(t)}{t^{1-c}}))'$$

Using the assumption that  $\frac{\omega(t)}{t^\alpha}$  is a decreasing function and choosing  $c = 1 - \alpha$  we obtain the desired inequality, proving Claim 4.  $\square$

Now we can apply Claim 4 to the desirable inequality:

$$C \cdot C' \cdot \Lambda'(\Lambda^{-1}(t)) \geq \omega(\phi_0^{-1}(e^{-t \cdot C} \cdot \phi_0(x_0))).$$

It is enough to show that

$$C \cdot C' \cdot c \cdot \omega(\frac{1}{\Lambda^{-1}(t)}) \geq \omega(\phi_0^{-1}(e^{-t \cdot C} \cdot \phi_0(x_0)))$$

We can choose  $C$  in a way that  $C \cdot C' \cdot c \in \mathbb{N}$ . Notice that

$$C \cdot C' \cdot c \cdot \omega(\frac{1}{C \cdot C' \cdot c} \cdot \phi_0^{-1}(e^{-C \cdot t} \cdot \phi_0(x_0))) \geq \omega(\phi_0^{-1}(e^{-C \cdot t} \cdot \phi_0(x_0)))$$

due to the sub-additivity property of  $\omega$ . Thus, it is enough to show that

$$C \cdot C' \cdot c \cdot \omega(\frac{1}{\Lambda^{-1}(t)}) \geq C \cdot C' \cdot c \cdot \omega(\frac{1}{C \cdot C' \cdot c} \cdot \phi_0^{-1}(e^{-C \cdot t} \cdot \phi_0(x_0))).$$

In other words,

$$C \cdot C' \cdot c \cdot \frac{1}{\Lambda^{-1}(t)} \geq \phi_0^{-1}(e^{-C \cdot t} \cdot \phi_0(x_0)).$$

Since  $\phi_0(x_0) < 1$  and  $\phi_0^{-1}$  is monotonically increasing, we can drop this multiplier and prove that

$$C \cdot C' \cdot c \cdot \frac{1}{\Lambda^{-1}(t)} \geq \phi_0^{-1}(e^{-C \cdot t}).$$

Denote by  $V = \frac{1}{\Lambda^{-1}(t)}$ . We have to prove the following inequality:

$$C \cdot C' \cdot c \cdot V \geq \phi_0^{-1}(e^{-C \cdot \Lambda(\frac{1}{V})}),$$

or, an equivalent form is

$$\phi_0(C \cdot C' \cdot c \cdot V) \geq e^{-C \cdot \Lambda(\frac{1}{V})} = e^{-C \cdot \frac{\omega(V)}{V}}$$

Note that

$$\phi_0(A \cdot V) = \int_0^{A \cdot V} \omega(t) dt \geq \int_{\frac{A \cdot V}{2}}^{A \cdot V} \omega(t) dt \geq \frac{A \cdot V}{2} \cdot \omega\left(\frac{A \cdot V}{2}\right).$$

Here, we used the monotonicity of  $\omega(x)$ . Hence, it is enough to show that:

$$\frac{C \cdot C' \cdot c \cdot V}{2} \cdot \omega\left(\frac{C \cdot C' \cdot c \cdot V}{2}\right) \geq e^{-C \cdot \frac{\omega(V)}{V}},$$

Equivalently,

$$\frac{C \cdot C' \cdot c \cdot V}{2} \cdot \omega\left(\frac{C \cdot C' \cdot c \cdot V}{2}\right) \cdot e^{C \cdot \frac{\omega(V)}{V}} \geq 1.$$

Since we can choose a constant  $C$  such that  $\frac{C \cdot C' \cdot c \cdot V}{2} \geq V$ , it is enough to prove that:

$$V \cdot \omega(V) \cdot e^{C \cdot \frac{\omega(V)}{V}} \geq 1.$$

Recall now, that  $\frac{\omega(x)}{x^\alpha}$  is decreasing in some  $[0, a(\alpha)]$ , thus there exists a constant  $c(\omega)$ , such that  $\omega(x) \geq c(\omega) \cdot x^\alpha$ . Substituting it in the above inequality, we have to show that

$$e^{\frac{C}{V^{1-\alpha}}} \geq \frac{1}{c(\omega) \cdot V^{1+\alpha}}$$

Certainly, such a  $C$  can be picked, since the left expression tends to  $\infty$  when  $V \rightarrow 0$ , while the right one tends to 0.  $\square$

The next lemma allows us to complete the proof of Theorem 5 for a general function  $f(x)$ .

**Lemma 5.** *Consider any  $x_0, x'_0 \in (0, 1)$  such that  $f(x_0) = f_0(x'_0)$  and  $\log \frac{\phi(x_0)}{\phi(x_n)} = |\log \frac{\phi(x_n)}{\phi(x_0)}|$  ( or in other words,  $\phi(x_0) > \phi(x_n)$  ), then*

$$\frac{\phi(x_0)}{\phi(x_n)} \leq C' \cdot \frac{\phi_0(x'_0)}{\phi_0(x'_{C \cdot n})}.$$

**Remark:** The same proof will work when we deal with the case  $\log \frac{\phi(x_n)}{\phi(x_0)} = |\log \frac{\phi(x_n)}{\phi(x_0)}|$ .

The proof of Lemma 5 is based on the following Lemma 6, whose proof will be provided in the end of this section.

**Lemma 6.** *Pick  $\alpha_0 > \dots > \alpha_m$ , such that  $\phi(\alpha_k) = \frac{\epsilon}{2^k}$  for all  $k = 0, \dots, m$ , and  $\phi_0(\beta_k) = \frac{\epsilon}{2^k}$  for  $k = 0, \dots, m$ . Then we have the following inequality*

$$\int_{\beta_m}^{\beta_0} \frac{dt}{\phi_0(t)} \leq C \cdot \int_{\alpha_m}^{\alpha_0} \frac{dt}{\phi(t)}.$$

*Proof of Lemma 5.*

**Claim 5.** *For fixed  $m \in \mathbb{N}$ , where  $\phi(x_0) = \phi_0(x'_0) = \epsilon$ , let  $n := n(m)$  and  $N := N(n)$  be such that*

$$\begin{aligned} \frac{\epsilon}{2^{m-1}} &\geq \phi(x_n) \geq \frac{\epsilon}{2^m} \quad , \\ \frac{\epsilon}{2^{m-2}} &\geq \phi_0(x'_N) \geq \frac{\epsilon}{2^{m-1}} \quad . \end{aligned}$$

*Then  $N \leq C \cdot n$ .*

*Proof of Claim 5.* We have  $n \geq c \cdot \int_{x_n}^{x_0} \frac{dt}{\phi(t)} \geq c \cdot \int_{\alpha}^{x_0} \frac{dt}{\phi(t)}$ , where  $\alpha \in [x_n, x_0]$  is such that  $\phi(\alpha) = \frac{\epsilon}{2^{m-1}}$ . Therefore

$$\int_{\alpha}^{x_0} \frac{dt}{\phi(t)} \geq C \cdot \int_{\phi_0^{-1}(\frac{\epsilon}{2^{m-1}})}^{x'_0} \frac{dt}{\phi_0(t)} \geq C \cdot \int_{x'_N}^{x'_0} \frac{dt}{\phi_0(t)} \geq C' \cdot N.$$

We have obtained that  $N \leq C \cdot n$ . This proves Claim 5.  $\square$

Now we provide details, which complete the proof of Lemma 5. We may reformulate the previous claim as follows. Let  $m \in \mathbb{N}$  some fixed number, if  $m-1 \leq \log \frac{\phi(x_0)}{\phi(x_n)} \leq m$  and  $m-2 \leq \log \frac{\phi_0(x'_0)}{\phi_0(x'_N)} \leq m-1$ , then  $N \leq C \cdot n$ . In particular, we have

$$\log \frac{\phi(x_0)}{\phi(x_n)} \leq \log \frac{\phi_0(x'_0)}{\phi_0(x'_N)} + 2.$$



The inequality  $N \leq C \cdot n$ , implies that  $0 < x'_{C \cdot n} \leq x'_N$  and  $\frac{1}{\phi_0(x'_N)} \leq \frac{1}{\phi_0(x'_{C \cdot n})}$ . Substituting it into  $\log \frac{\phi(x_0)}{\phi(x_n)} \leq \log \frac{\phi_0(x'_0)}{\phi_0(x'_N)} + 2$ , we obtain

$$\log \frac{\phi(x_0)}{\phi(x_n)} \leq \log \frac{\phi_0(x'_0)}{\phi_0(x'_{c \cdot n})} + 2,$$

or

$$\frac{\phi(x_0)}{\phi(x_n)} \leq C' \cdot \frac{\phi_0(x'_0)}{\phi_0(x'_{c \cdot n})}.$$

The proof of Lemma 5 is completed.  $\square$

Now, let us complete the proof of Theorem 5 (modulo Lemma 6). Combining Lemmas 4 and 5 we have that

$$|\log \frac{\phi(x_{n+1})}{\phi(x_1)}| \leq \log C' + \log \frac{\phi_0(x'_0)}{\phi_0(x'_{c \cdot (n+1)})}$$

$$\leq \log C' + Cc(n+1)\omega\left(\frac{1}{c \cdot (n+1)}\right) \leq C''n\omega\left(\frac{1}{n}\right)$$

, since we have  $\frac{\omega(x)}{x} \rightarrow \infty, x \rightarrow 0$ , due to the assumptions of the theorem. Therefore, due to Lemma 3,

$$|\log (f^n)'(x)| \leq |\log \frac{\phi(x_{n+1})}{\phi(x_1)}| + C \cdot n\omega\left(\frac{1}{n}\right) \leq C''' \cdot n\omega\left(\frac{1}{n}\right)$$

The proof of Theorem 5 is completed up to the proof of Lemma 6 which will be provided now.

*Proof of Lemma 6.* The proof is based on the following series of claims:

**Claim 6.** Suppose that  $\phi(s) = \phi_0(s_0)$ . Then

$$\phi'(s) \leq 2 \cdot \phi'_0(s_0).$$

*Proof of Claim 6.* Note that  $\phi'(x) \geq \phi'(a) - \omega(a - x)$ . Integrate both sides and obtain the following:

$$\int_t^a \phi'(x)dx \geq \int_t^a (\phi'(a) - \omega(a - x))dx,$$

where  $t < a$ . Thus we have

$$\phi(a) - \phi(t) \geq (a - t)\phi'(a) - \phi_0(a - t),$$

equivalently

$$\frac{\phi(a) - \phi(t)}{a - t} \geq \phi'(a) - \frac{\phi_0(a - t)}{a - t}.$$

The inequality  $|\phi'(t)| \leq \omega(t)$  implies that  $\phi(x) \leq \int_0^x |\phi'(t)| dt \leq x \cdot \omega(x)$ . Pick  $a = s$  and  $t = s - s_0$  ( $s > s_0$ , since  $\phi_0(x) \geq \phi(x)$ ), we obtain that:

$$\phi'(s) \leq \frac{\phi(s) - \phi(s - s_0)}{s_0} + \frac{\phi(s_0)}{s_0} \leq 2 \cdot \frac{\phi_0(s_0)}{s_0} \leq 2 \cdot \omega(s_0)$$

□

**Claim 7.** If  $\phi_0(\beta_0) = \phi(\beta) = \epsilon$  and  $\phi_0(\alpha_0) = \phi(\alpha) = \frac{\epsilon}{2}$ , then,

$$\beta - \alpha \geq \frac{1}{2} \cdot (\beta_0 - \alpha_0).$$

*Proof of Claim 7.* Define  $g(x) = \phi_0^{-1}(\phi(x))$  and notice that

$$g'(x) = \frac{\phi'(x)}{\phi'_0(\phi_0^{-1}(\phi(x)))} \leq 2,$$

this follows from Claim 6. Now we conclude that  $g(x)$  is a Lipschitz, namely,  $|\phi_0^{-1}(\phi(x)) - \phi_0^{-1}(\phi(y))| \leq 2 \cdot |x - y|$ , in particular,  $|g(\beta) - g(\alpha)| \leq 2 \cdot |\beta - \alpha|$ , where  $\beta_0 = \phi_0^{-1}(\phi(\beta)) = g(\beta)$  and  $\alpha_0 = \phi_0^{-1}(\phi(\alpha)) = g(\alpha)$ . □

Now let  $x_0, x'_0 \in (0, 1)$  be such that  $\phi(x_0) = \phi_0(x'_0) = \epsilon$ , denote by  $a_0 = x_0$ ,  $b_0 = x'_0$  and define the values  $a_k$  and  $b_k$ ,  $k = 1, \dots, m$ , by

$$a_k = \max\{x < a_{k-1} : \phi(x) = \frac{\epsilon}{2^k}\}$$

$$\phi_0(b_k) = \frac{\epsilon}{2^k}.$$

Choose  $a'_{k+1} \in [a_k, a_{k+1}]$  such that  $\phi(a'_{k+1}) = \frac{\epsilon}{2^k}$  and  $\frac{\epsilon}{2^{k+1}} \leq \phi(x) \leq \frac{\epsilon}{2^k}$  for all  $x \in (a_{k+1}, a'_{k+1})$ .

**Claim 8.**

$$\int_{a_m}^{a_0} \frac{dt}{\phi(t)} \geq C \cdot \int_{b_m}^{b_0} \frac{dt}{\phi_0(t)}$$

*Proof of Claim 8.* We will split both integrals to the sums:

$$\int_{a_m}^{a_0} \frac{dt}{\phi(t)} = \sum_{j=0}^{m-1} \int_{a_{j+1}}^{a_j} \frac{dt}{\phi(t)},$$

and,

$$\int_{b_m}^{b_0} \frac{dt}{\phi_0(t)} = \sum_{j=0}^{m-1} \int_{b_{j+1}}^{b_j} \frac{dt}{\phi_0(t)},$$

hence it is enough to check that

$$\int_{a_{j+1}}^{a_j} \frac{dt}{\phi(t)} \geq C \cdot \int_{b_{j+1}}^{b_j} \frac{dt}{\phi_0(t)}.$$

Indeed,

$$\int_{a_{j+1}}^{a_j} \frac{dt}{\phi(t)} \geq \int_{a_{j+1}}^{a'_{j+1}} \frac{dt}{\phi(t)} \geq (a'_{j+1} - a_{j+1}) \cdot \frac{2^j}{\epsilon}.$$

Using Claim 7 we obtain that

$$(a'_{j+1} - a_{j+1}) \cdot \frac{2^j}{\epsilon} \geq \frac{1}{4} \cdot (b_j - b_{j+1}) \cdot \frac{2^{j+1}}{\epsilon} \geq \frac{1}{4} \cdot \int_{b_{j+1}}^{b_j} \frac{dt}{\phi_0(t)}.$$

The claim is proved.  $\square$

Now consider an arbitrary  $\alpha_0 := a_0$ , and pick  $\alpha_0 < \alpha_1 < \dots < \alpha_m$  such that  $\phi(\alpha_k) = \frac{\epsilon}{2^k}$ , for all  $k = 0, \dots, m$ . Hence, by using Claim 6, we have

$$\int_{\alpha_m}^{\alpha_0} \frac{dt}{\phi(t)} \geq \int_{a_m}^{a_0} \frac{dt}{\phi(t)} \geq C \cdot \int_{\beta_m}^{\beta_0} \frac{dt}{\phi_0(t)},$$

Proving Lemma 6.  $\square$

It completes the proof of Theorem 6.  $\square$

### 3. SHARPNESS: PROOFS OF THEOREMS 6 AND 7

*Proof of Theorem 6.* Define  $\phi(x) = \int_0^x \omega(t)dt$  and  $f(x) = x - \phi(x)$ , in some interval  $[0, \epsilon]$ . Extend  $f(x)$  arbitrarily  $C^\infty$ -smoothly to the whole interval  $[0, 1]$  in such way that  $f(1) = f'(1) = 1$ . We work in the interval  $[0, \epsilon]$ . Denote by  $h(x) := \frac{\omega(x)}{x}$ . By Lemma 2,

$$(\Delta) \quad \log\left(\frac{1}{\phi(x_n)}\right) - C \cdot n \cdot \omega\left(\frac{1}{n}\right) \leq \log(f^n)'(x_1).$$

Now, we estimate from below the left hand side of  $(\Delta)$ .

**Claim 9.**

$$\omega(x_n) \leq \frac{C}{n}$$

*Proof.* Recall that,

$$\frac{t}{4} \cdot \omega(t) \leq \frac{t}{2} \cdot \omega\left(\frac{t}{2}\right) \leq \int_{\frac{t}{2}}^t \omega(t) \leq \phi(t) = \int_0^t \omega(t) \leq t \cdot \omega(t).$$

We use this observation in the following form:

$$c' \cdot \int_{x_n}^{x_1} \frac{dt}{t^2 \cdot h(t)} \leq c \cdot \int_{x_n}^{x_1} \frac{dt}{\phi(t)} \leq n \leq C \cdot \int_{x_n}^{x_1} \frac{dt}{\phi(t)} \leq C' \cdot \int_{x_n}^{x_1} \frac{dt}{t^2 \cdot h(t)}$$

We make change of variables  $[s = \frac{1}{t}]$  and get that

$$c' \cdot \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})} \leq n \leq C' \cdot \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})},$$

we obtain using integration by parts,

$$= C' \cdot \left( \frac{1}{x_n \cdot h(x_n)} - \frac{1}{x_1 \cdot h(x_1)} - \int_{\frac{1}{x_n}}^{\frac{1}{x_1}} \frac{h'(\frac{1}{s})}{sh^2(\frac{1}{s})} ds \right).$$

Recall that  $\frac{\omega(x)}{x^\alpha} = x^{1-\alpha} \cdot h(x)$  is an increasing function for all  $0 < \alpha < 1$  in the corresponding intervals  $[0, a(\alpha)]$ , while  $h'(x) < 0$  for all  $x \in [0, 1]$ , since  $h(x)$  is decreasing. In particular,

$$\left( \frac{\omega(x)}{x^\alpha} \right)' = (x^{1-\alpha} \cdot h(x))' = (1 - \alpha) \cdot x^{-\alpha} \cdot h(x) + x^{1-\alpha} \cdot h'(x) > 0,$$

for  $x \in [0, a(\alpha)]$ . We multiply the last inequality by  $x^\alpha$  and get that:

$$(1 - \alpha)h(x) \geq -x \cdot h'(x).$$

Let us pick  $\alpha$  which is very close to 1. We can assume that  $x_1 \in [0, a(\alpha)]$ , what implies that all iterations are also in  $[0, a(\alpha)]$ , namely  $x_n \in [0, a(\alpha)]$ . We get the following estimate

$$\int_{\frac{1}{x_n}}^{\frac{1}{x_1}} \frac{-h'(\frac{1}{s})}{sh^2(\frac{1}{s})} ds \leq (1 - \alpha) \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})}.$$

Therefore,

$$c' \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})} \leq n \leq C' \left( \frac{1}{x_n \cdot h(x_n)} - \frac{1}{x_1 \cdot h(x_1)} \right) + (1 - \alpha) \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})}$$

Recalling that  $xh(x) = \omega(x)$ , we get

$$(c' - (1 - \alpha) \cdot C') \cdot \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})} \leq C' \cdot \left( \frac{1}{\omega(x_n)} - \frac{1}{\omega(x_1)} \right).$$

For  $\alpha$  very close to 1, we have

$$\frac{c'}{2} \cdot \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})} \leq (c' - (1 - \alpha) \cdot C') \cdot \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})},$$

hence, we have

$$\frac{c'}{2} \cdot \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})} \leq C' \cdot \left( \frac{1}{\omega(x_n)} - \frac{1}{\omega(x_1)} \right).$$

Therefore,

$$n \leq C' \cdot \int_{\frac{1}{x_1}}^{\frac{1}{x_n}} \frac{ds}{h(\frac{1}{s})} \leq \frac{2(C')^2}{c'} \cdot \left( \frac{1}{\omega(x_n)} - \frac{1}{\omega(x_1)} \right),$$

whence,

$$\frac{c'}{2(C')^2} \omega(x_n) \leq \frac{1}{n}.$$

It completes the proof of Claim 9.  $\square$

Recall that  $\phi(x) \leq x\omega(x)$ . Due to Claim 9 and the monotonicity of  $\omega^{-1}$ , we have

$$\phi(x_n) \leq x_n \omega(x_n) \leq \frac{a}{n} \cdot \omega^{-1}\left(\frac{a}{n}\right).$$

Therefore,

$$\frac{n}{a \cdot \omega^{-1}\left(\frac{a}{n}\right)} \leq \frac{1}{\phi(x_n)}.$$

Substituting it into  $(\Delta)$ , we have

$$\log \frac{n}{a \cdot \omega^{-1}\left(\frac{a}{n}\right)} - C \cdot n \cdot \omega\left(\frac{1}{n}\right) \leq \log(f^n)'(x_1).$$

Consider any  $\varepsilon > 0$ , let us check that

$$(1 - \varepsilon) \cdot \log \frac{n}{\omega^{-1}\left(\frac{\varepsilon}{n}\right)} \leq \log\left(\frac{n}{a \cdot \omega^{-1}\left(\frac{a}{n}\right)}\right) - C \cdot n \cdot \omega\left(\frac{1}{n}\right),$$

when  $n \rightarrow \infty$ . That is equivalent to

$$\frac{Cn\omega\left(\frac{1}{n}\right)}{\log \frac{n}{a\omega^{-1}\left(\frac{a}{n}\right)}} \leq \varepsilon,$$

as  $n \rightarrow \infty$ . Indeed,

$$\frac{Cn\omega\left(\frac{1}{n}\right)}{\log \frac{n}{a\omega^{-1}\left(\frac{a}{n}\right)}} \leq \frac{Cn\omega\left(\frac{1}{n}\right)}{\log \frac{n^2}{a^2}} \leq \frac{C}{2} \cdot \frac{\omega\left(\frac{1}{n}\right)}{\frac{\log n}{n}} \rightarrow 0,$$

here we used  $\omega(x) \leq x$  and that  $\lim_{x \rightarrow 0} \frac{\omega(x)}{x} = 0$ . It completes the proof of Theorem 6.  $\square$

*Proof of Theorem 7.* The proof is based on the construction presented in [B]. Let  $0 < \varepsilon < 1$  be an arbitrary number, define

$$\phi_\varepsilon(x) = x - \left(1 + \frac{1}{x}\right)^{-1} - x^{2+\varepsilon} \cdot \omega(x) \cdot \sin\left(\frac{2\pi}{x}\right)$$

$$f_\varepsilon(x) = x - \phi_\varepsilon(x)$$

on some interval  $[0, a(\varepsilon)]$ . Note that  $f_\varepsilon(0) = 0$ ,  $f'_\varepsilon(0) = 1$  and for  $0 < k^{-1} < a(\varepsilon)$ ,  $f_\varepsilon(k^{-1}) = (k+1)^{-1}$ . It is possible to choose  $a(\varepsilon)$  in a way that

1.  $f'_\varepsilon(x) > 0$  for all  $x \in [0, a(\varepsilon)]$ .

2.  $f_\varepsilon(x)$  does not admit any fixed points in  $(0, a(\varepsilon)]$ .
3. The following inequality is satisfied

$$|f'_\varepsilon(x) - f'_\varepsilon(y)| \leq C \cdot \omega(|x - y|),$$

for all  $x, y \in [0, a(\varepsilon)]$ , where  $C$  is an absolute constant, which does not depend on  $\varepsilon$ .

Then, for  $0 < k^{-1} < a(\varepsilon)$ ,

$$\begin{aligned} \log(f_\varepsilon^N)'(k^{-1}) &= \sum_{j=0}^{N-1} \log f'_\varepsilon\left(\frac{1}{k+j}\right) \\ &\geq \sum_{j=0}^{N-1} \log(((k+j)^{-2} + 1)^{-2} + (k+j)^{-\varepsilon} \cdot \omega((k+j)^{-1})) \\ &\geq c'(\varepsilon) \cdot N \cdot (k+N-1)^{-\varepsilon} \cdot \omega((k+N-1)^{-1}) \\ &\geq c(\varepsilon) \cdot N^{1-\varepsilon} \cdot \omega(N^{-1}), \end{aligned}$$

as  $N \rightarrow \infty$ .

We are going to construct a diffeomorphism  $f \in \text{Diff}_0^\omega[0, 1]$ , which will be composed of a suitable pasting of the frame functions  $f_\varepsilon$ .

Let  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  be an arbitrary monotonically decreasing sequence of reals numbers which tends to 0. Pick two sequences  $\{a_k\}_{k \in \mathbb{N}}$ ,  $\{b_k\}_{k \in \mathbb{N}}$  monotonically decreasing sequences of real numbers which tend to 0, such that  $a_k > b_{k+1}$ , for all  $k \in \mathbb{N}$ . Define now

$$\tilde{f}_{\varepsilon_k}(x) = \begin{cases} f_{\varepsilon_k}(x - a_k), & x \in [a_k, a_k + a(\varepsilon_k)] \\ \Psi_k(x), & x \in [a_k + a(\varepsilon_k), b_k] \end{cases}$$

where  $\Psi_k(x)$  is a monotonic  $C^\infty$ -continuation of  $f_{\varepsilon_k}(x - a_k)$  to the whole interval  $[a_k, b_k]$ , without fixed points on the interval  $[a_k + a(\varepsilon_k), b_k]$  with the property  $\Psi_k(b_k) = b_k$ ,  $\Psi'_k(b_k) = 1$ , and with bounded second derivative  $|\Psi''_k(x)| < 1$ . Define

$$f(x) = \begin{cases} \tilde{f}_{\varepsilon_k}(x), & x \in [a_k, b_k] \\ x, & x \in [0, 1] \setminus \cup_{k \in \mathbb{N}} [a_k, b_k] \end{cases}$$

Since  $\Psi_k(x)$  is  $C^\infty$  with second bounded derivative, it is not hard see that  $|f'(x) - f'(y)| \leq C(f)\omega(|x - y|)$ , for all  $x, y \in [0, 1]$ .

Now, choose an arbitrary  $\varepsilon > 0$ , there exists  $\varepsilon_k < \varepsilon$ . Pick any  $m^{-1} < a(\varepsilon_k)$ . Thus, we have

$$\begin{aligned} \log \Gamma_N(f) &\geq \log (f^N)'(a_k + m^{-1}) = \log (f_{\varepsilon_k}^N)'(m^{-1}) \\ &\geq c(\varepsilon_k) \cdot N^{1-\varepsilon_k} \omega(N^{-1}) \geq c(\varepsilon) \cdot N^{1-\varepsilon} \omega(N^{-1}). \end{aligned}$$

Theorem 7 is proved.  $\square$

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